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Lattice chirality and the decoupling of mirror fermions

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Abstract: We show, using exact lattice chirality, that partition functions of lattice gauge theories with vectorlike fermion representations can be split into "light" and "mirror" parts, such that the "light" and "mirror" representations are chiral. The splitting of the full partition function into "light" and "mirror" is well defined only if the two sectors are separately anomaly free. We show that only then is the generating functional, and hence the spectrum, of the mirror theory a smooth function of the gauge field background. This explains how ideas to use additional non-gauge, high-scale mirror-sector dynamics to decouple the mirror fermions without breaking the gauge symmetry — for example, in symmetric phases at strong mirror Yukawa coupling — are forced to respect the anomaly-free condition when combined with the exact lattice chiral symmetry. Our results are also useful in explaining a paradox posed by a recent numerical study of the mirror-fermion spectrum in a toy would-be-anomalous two-dimensional theory. In passing, we prove some general properties of the partition functions of arbitrary chiral theories on the lattice that should be of interest for further studies in this field.

Keywords: Gauge Symmetry, Anomalies in Field and String Theories, Lattice Gauge Field Theories, Strong Coupling Expansion.

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1. Introduction and summary

1.1 Motivation

The study of strong-coupling chiral gauge dynamics is an outstanding problem of great interest, both on its own and for its possible relevance to phenomenology. Whereas the standard model of elementary particle physics is a weakly coupled chiral gauge theory, additional strong chiral gauge dynamics at (multi-) TeV scales may be responsible for breaking the electroweak symmetry and fermion mass generation.

Several different approaches are currently available for the study of the strong-coupling behavior of chiral gauge theories. Notably, one has 't Hooft's anomaly matching and most attractive channel arguments, which are complemented by the "power of holomorphy" in supersymmetric theories. Scaling arguments and effective NJL-like models, both using results from QCD as a stepping stone, have also been employed extensively; on the other hand, large-N expansions, including the recently considered gravity duals in the AdS/CFT (AdS/QCD) framework, do not usefully apply to chiral gauge theories. None of these approaches represents a "first principles" method, with an accuracy that can be systematically improved. The space-time lattice regularization remains, to this day, the only way offering hope for such a systematic progress.

During the past two decades, since the work of Ginsparg and Wilson (GW) [1], there has been significant progress in understanding chiral symmetries on the lattice [2]–[7]; further references are given in the reviews [8, 9], while [10 – 12] contain more recent work.

Recently, the existence of an exactly gauge invariant[1] lattice construction of anomaly-free chiral gauge theories using exact lattice chiral symmetries has been proven in several particular cases [9, 13]. However, an explicit formulation of the action and measure outside of perturbation theory is currently not available. We thus believe that the further study of the problem, the consideration of new proposals, and of their relationship to old and new advances in the field is a worthwhile task.

1.2 What this paper is about

In essence, this paper is about revisiting an old idea [21] in light of the new understanding of exact lattice chirality. The idea is to begin with a vectorlike theory, whose explicit lattice formulation poses no problems. The chiral components of the vectorlike fermions are split into "light" and "mirror" fermions. The "light" fermions have the chirality and group representations of the desired target chiral gauge theory and the "mirrors" are simply their opposite-chirality partners. The vectorlike theory is then deformed in a way that (ideally) only affects the "mirror" fermions: for example, one adds appropriate Yukawa interactions, or four- and multi-fermion interactions. The goal of the deformation is to ensure that, when the parameters of the deformation are chosen appropriately, the mirror fermions decouple without breaking the gauge symmetry. Thus, at low energies, the desired unbroken chiral gauge theory is supposed to emerge.

The decoupling of "mirror" fermions in chiral representations without breaking the chiral symmetry (which is gauged in chiral gauge theories) is possible in the so-called strong-coupling symmetric phases of lattice Yukawa [24 – 27] or multi-fermion-interaction [21] theories (earlier, the possibility of nonzero fermion masses without chiral symmetry breaking has been discussed, for two dimensional models, in [28]).

From the continuum physics point of view, the strong-coupling symmetric phases are a lattice artifact. Their existence, in either two or four dimensions, is established using the lattice strong-coupling expansion, where all correlations have a range smaller than the lattice spacing. One can thus say that in models with a strong-coupling symmetric phase and heavy mirrors, their mass is "higher than the ultraviolet cutoff"—the physics at high scales being that of lattice particles with small site-to-site hopping probability. The strong-

[1]There also exists a point of view [14] that an exact gauge invariance at finite lattice spacing may not be necessary and that "gauge averaging" of the fermion determinant will wash out, by a mechanism due to [15, 16], any gauge-breaking effects in the continuum limit in the anomaly-free case. While some numerical evidence supports this view [17, 18], the issue is far from settled, see [19] and the review [8]. For other ideas giving up exact gauge invariance, see [20, 10].

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coupling symmetric phases of lattice Yukawa or multi-fermion interaction models are thus analogous to the well-known high-temperature disordered (hence symmetric) phases of spin systems.

For the continuum physicist, who is unlikely to proceed past this Introduction, we will now give a cartoon-like continuum description of the physics. This will also serve to illustrate the idea behind using strong interactions to decouple the mirrors and help us state the main issues we would like to address.

Consider thus the classic example of a four-dimensional chiral gauge theory with non-trivial strong-coupling dynamics — see, e.g., [29]—an SU(5) gauge theory with a $\mathbf{5}^*$ and a $\mathbf{10}$ Weyl fermion representation. We use two-component spinor notation to describe the desired "light" fermions:

$$\psi^i_\alpha \sim \mathbf{5}^*, \quad \chi_{ij\,\alpha} \sim \mathbf{10}\,, \quad \zeta_\alpha \sim \mathbf{1}\,, \tag{1.1}$$

(here i denotes an SU(5) (anti-)fundamental and $\alpha = 1,2$, an SL$(2,C)$ index) and their "mirror" partners:

$$\eta_{i\alpha} \sim \mathbf{5}, \quad \rho^{ij}_\alpha \sim \mathbf{10}^*\,, \quad \xi_\alpha \sim \mathbf{1}\;. \tag{1.2}$$

In this notation a Dirac mass term for the $\mathbf{5}$ would be $\psi^{i\alpha}\eta_{i\alpha}$+h.c.. The gauge singlet Dirac fermion (with Weyl components ζ, ξ) is a field whose ξ component will play an important role in the strong mirror dynamics; an entire singlet Dirac multiplet was added to make sure the fermion representation (1.1), (1.2) is vectorlike and thus easy to put on the lattice.

The target SU(5) chiral gauge theory has one anomaly-free U(1) global symmetry, under which ψ^i has charge -3 and χ_{ij} charge 1; on the other hand, the vectorlike theory with fermion content (1.1), (1.2) has more exact global symmetries. Now, to decouple the mirrors (1.3), one adds interactions involving (ideally) only the mirror fields, of the form:

$$\lambda\, \xi^\alpha\, \eta^i_\alpha \eta^{j\,\beta} \rho_{ij\,\beta} + \ldots \tag{1.3}$$

where the dots denote terms needed to break the extra global symmetries.

The main insight helping to decouple the mirrors is the realization that the strong lattice four-fermi interaction (1.3) can lead to the formation of SU(5) invariant mirror composite states, which can acquire mass without breaking SU(5). We stress again that the strong-coupling symmetric phase and the formation of the singlet mirror composite states requires $\lambda \gg 1$ in UV-cutoff units; this only makes sense on the lattice, and the spectrum can be studied using the strong-coupling expansion (for details, see the appendix of ref. [21]). For example, a possible composite of the mirror fermions is the $\eta\eta\rho$ ($\mathbf{5}$-$\mathbf{5}$-$\mathbf{10}^*$) invariant appearing in (1.3). It can acquire a large Dirac mass by pairing with the singlet mirror field ξ and can thus decouple from the low-energy physics without breaking the SU(5) symmetry; at strong coupling all mirror fermions are similarly bound in massive composites and decouple.[2] Since the SU(5) gauge interaction is asymptotically free, the

[2]Ideas involving strong-Yukawa symmetric phases work similarly [24 – 27] and are closely related to the multi-fermion interaction ones [32].

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strong mirror dynamics at and above the cutoff scale should have a parametrically small effect on the infrared chiral dynamics. Thus, the desired unbroken chiral gauge theory with massless fermion spectrum (1.1) is recovered in the infrared.

The idea to decouple the mirrors in the way described above is attractive and many important advances in understanding the strong-coupling symmetric phases in both Yukawa and multi-fermion-interaction theories were made in the past. However, in all cases studied, the spectrum of massless fermion states (when they existed) was found to be vectorlike:

(i) The most notable reason is the fact that on the lattice — until the recent advances in exact lattice chirality — there was no way to define chiral components of the spinors at finite lattice spacing while avoiding fermion doubling. Because chiral symmetries are broken on the lattice with the traditional Wilson formulation, the strong interactions (1.3) were, in all cases, also "felt" by the "light" fermions, causing either the "light" fermions to obtain mass or the "mirror" fermions to become massless; see [30 – 32].

(ii) Moreover, since the SU(5) gauge dynamics is only a spectator of the strong interactions whose purpose is to decouple the mirrors, it is not clear how the non-gauge strong interactions of the mirror sector were supposed to "know" about chiral gauge anomalies and how they would decide to "enforce" the anomaly-freedom requirement on the light chiral spectrum or else, break the gauge symmetry [35 – 38].[3]

In this paper, we will address the above issues by using the exact chirality-preserving GW-fermion formulation of lattice vectorlike theories via the Neuberger-Dirac operator:

- It is well known that the GW-fermion formulation allows one to define chiral components of the spinors at finite lattice spacing without introducing doublers. The "light" chiral components can be then excluded from participating in the strong "mirror" interactions, like the one in (1.3). The lattice theory can then be arranged to have exactly the global symmetries and anomalies of the target continuum chiral theory, something that earlier Yukawa or four-fermi proposals could not achieve [12].

- Furthermore, by considering in detail the split of the vectorlike lattice partition function into "light" and "mirror" parts in an arbitrary gauge background, we will show that the anomaly-free condition on the light spectrum is also enforced by consistency of the GW formulation of lattice chirality. We will make extensive use of the work of Neuberger [6] and Lüscher [13] on chiral anomalies in the overlap/GW-fermion formalism, see also [33]

The proposal to use strong-coupling Yukawa models with GW fermions to decouple the mirrors in a vectorlike gauge theory was made in [12], where the many desirable features of such a formulation were pointed out.[4] The proposal is attractive, as it gives an explicit

[3] For example, ref. [32] found that in the model of [21] the non-gauge mirror dynamics was essentially the same in models with anomaly-free and anomalous fermion spectrum.

[4] We note that ref. [34] made an earlier suggestion along similar lines, in the framework of a domain wall with a finite fifth dimension.

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gauge-invariant definition of the measure and lattice action, and because it has all the right symmetries and anomalies of the target chiral gauge theory already at finite lattice spacing. However, this elegance comes at a price — the study of the mirror dynamics at strong coupling, which is needed to show that the mirrors do indeed decouple, is complicated by the exponential-only locality of the Neuberger-Dirac operator [39, 40]. The strong-coupling expansions used in relatively simple models [21, 24 – 27, 30 – 32] to predict the formation of heavy fermion composites without breaking the chiral symmetry are not easy to implement and a Monte-Carlo study is called for.

A numerical study of the strong-coupling mirror dynamics of a toy two-dimesional model was performed in [41], for a vanishing gauge background. The numerical evidence found there indicates that, indeed, the mirror sector decouples at strong mirror Yukawa coupling. The questions of gauge anomalies in the light target theory and the ways the dynamics would prevent them was not addressed. This is the main issue we focus on in this paper.

1.3 Outline and summary of results

Much of the discussion in this paper is rather technical. Here we outline the main points and summarize our results. The reader is assumed to be familiar with the GW relation and the exact lattice chiral symmetry in vectorlike theories; for a review, see [9] and references therein (our notation for the Neuberger-Dirac operator, the modified-γ_5 projectors, and their eigenvectors is established in section 2.1).

In section 2.2, we consider in detail how the partition function of a vectorlike theory splits into left- and right- chirality components, using the eigenvectors of the modified-γ_5 as basis. We also explicitly work out the transformations of the left- and right- chirality partition functions and of the Jacobian under changes of the gauge background.

In section 3, we turn to the description of what we call the "1-0" model: a toy two dimensional model, used in a Monte-Carlo study of the decoupling of the mirrors in the strong-Yukawa symmetric phase [41]. We show how the partition function of this model (with vectorlike fermion content) splits into a "light" and "mirror" part in an arbitrary gauge background. Only the "mirror" degrees of freedom participate in the strong Yukawa interaction, which is introduced to decouple the "mirrors" from the long-distance physics (similar in spirit to (1.3)).

Using the results of section 2.2, we then work out the gauge transformations of the "light" and "mirror" partition functions and show that the gauge transformation of the "mirror" partition function precisely cancels the anomaly of the light fermions, independent of the value of the mirror Yukawa coupling(s) and for arbitrary gauge backgrounds.

We then contrast this finding with the numerical results of [41]. The Monte-Carlo simulation of the mirror dynamics at strong Yukawa coupling and in vanishing gauge background provided evidence for the decoupling of the mirror sector without breaking the gauge symmetry (i.e., of the existence of the desired strong-Yukawa symmetric phase with heavy mirrors). The massless spectrum of the theory consists of a left-handed fermion of unit charge and a right-handed singlet under the gauge group. These numerical results, combined with the exact gauge transforms of the mirror partition functions worked out

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above, present us with a paradox. If the decoupling of the mirrors at strong Yukawa coupling and zero gauge background persists also for an infinitesimal gauge background, as one would naively expect based on "continuity", it is not clear how the heavy degrees of freedom could conspire to cancel the anomaly of the massless fermions. (see also the addendum for more discussion.)

As already alluded to, the resolution of the paradox is in the assumption of the continuity. It turns out that the "light"−"mirror" split of the partition function of the vectorlike theory is only well-defined if the light and mirror representation are separately anomaly-free. The results of the Monte-Carlo simulations in the "1-0" model hold for the trivial ($U = 1$) gauge background. However, we will show that the mirror partition function is not a smooth function of the gauge background precisely at $U = 1$ and that this singularity prevents any discussion of the mirror spectrum in general backgrounds.

We explain in detail how this comes about in sections 4 and 5. There, while still with the "1-0" model in mind, we switch the focus of our discussion to the most general form of chiral theories and study their properties based only on their defining characteristics. Many results found there are therefore of general applicability and should be important for further studies of chiral theories on the lattice.

We begin, in section 4.1 with a discussion of the dependence of the modified-chirality basis vectors on the gauge-field background; the material of this section is known, but for completeness we present a short self-contained derivation.

In section 4.2, we prove our main result on the variation of the most general chiral partition function under changes of the gauge background. We show that the variation of the partition function defined with an arbitrary chiral action always factorizes into a variation that depends on the basis vectors and a variation only due to the dependence of the operators included in the action on the gauge background. This generalizes the known results for simple actions; see section (2.2). It is an important piece of knowledge since it isolates the anomalies from the details of the chiral theory and manifestly realizes on the lattice the idea that anomalies are determined only by the representation of the fields and not by the details of the Lagrangian. It has at least a few surprisingly powerful implications, one of which will be explained in section 5.

In section 4.3 and 4.3.1 we explain how, in the case of an anomalous representation, the chiral fermion measure can not be defined as a smooth function of the gauge background. We use the Wilson-line subspace of the gauge field background to illustrate, following [6], the topological obstruction of defining a smooth fermion measure due to the anomaly. We explicitly show that in our "1-0" toy model the mirror-fermion measure is not a smooth function of the gauge fields exactly at vanishing gauge background. We then explicitly demonstrate (within the Wilson-line subspace only) how to construct smooth fermion measures in the case of anomaly-free representations, for example, in the "3-4-5" model [12] by showing how the singularities due to different representations can cancel each other and how the phase ambiguity of the chiral partition function enters to help.

Finally, in section 5, we consider an interesting application of the results proven in section 4.2 to show that the generating functional of the mirror theory is indeed a smooth function of the gauge background as long as the mirror representation is anomaly free.

Thus, as an encouraging message of this paper, one expects that a demonstration of the decoupling of the mirror sector in an anomaly free model at vanishing gauge background will persist, by smoothness, also for small gauge background, e.g., in perturbation theory with respect to the gauge coupling. The proof given in this section is also a general result independent on the details of the mirror action, and therefore the conclusion found there remains true for any well-behaved chiral theory as long as the anomaly-free condition is satisfied.

1.4 Outlook

The main statement we make in this paper is that the splitting of vectorlike partition functions into "light" and "mirror" parts is smooth, as a function of the gauge field background, only if the mirror and light fermion representations are separately anomaly free. This is a comforting result as it explains how the non-gauge dynamics introduced to decouple the mirrors is forced to obey the anomaly free condition, if one wishes to generalize the results to a full theory with dynamical gauge fields. If the gauge field is taken as fixed external background only, there might also exist other mechanisms that force the anomaly cancellation conditions as explained in the addendum.

We think that this result encourages further study of the decoupling of the mirror fermions in anomaly-free representations via strong lattice-cutoff-scale dynamics, such as that of strong-Yukawa symmetric phases. The next most important question is, of course, to demonstrate that the strong mirror dynamics does indeed cause the mirrors to decouple in anomaly-free cases and for trivial gauge background.

We stress the main advantages of the approach studied here: the fermion measure is well defined (as it is the trivial measure of the vectorlike theory), the global symmetries are realized exactly as in the desired target theory, and the partition function is exactly gauge invariant. Symmetry and beauty aside, the ultimate goal of the approach is to be useful for actual numerical simulations of chiral gauge theories. Whether this will happen depends on many yet unknown factors, notably the possible complexity or sign problem of the partition function. Here we only note that, in zero gauge background, the partition function of the "1-0" model at infinite Yukawa coupling was found in [41] to be real and positive; this raises hopes that in the theory with dynamical gauge fields the phase problem may be not too severe at large values of the Yukawa coupling. This issue certainly deserves more attention.

Finally, as already mentioned, the analytic strong-coupling expansion using the Neuberger-Dirac operator is complicated by its exponential-only locality [39, 40], leading one to suspect that Monte-Carlo simulations may appear as the only tool to study the strong-coupling mirror dynamics. However, we note the recent work [45] on an analytic strong-coupling expansion in some four-dimensional Yukawa models with GW fermions (at vanishing gauge background). Within the approximations used, analytic evidence — backed up by results of recent Monte-Carlo simulations [46]—for the existence of a strong-coupling symmetric phase was found. It may thus be interesting to study the possible application of these methods to models designed to decouple mirror fermions.

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2. Splitting partition functions of vectorlike theories into chiral components

2.1 Notations and basis vectors

In terms of the massive Wilson operator D_W, the modified-γ_5 matrix $\hat{\gamma}_5$ and the Neuberger-Dirac operator D are expressed as [9]:

$$\hat{\gamma}_5 = \frac{\gamma_5 A}{\sqrt{(\gamma_5 A)^2}}, \quad A \equiv 1 - D_W, \quad D \equiv 1 - \gamma_5 \hat{\gamma}_5 \,, \tag{2.1}$$

where D transforms covariantly under gauge transforms, $D_{xy} \to e^{i\omega_x} D_{xy} e^{-i\omega_y}$ and the Ginsparg-Wilson (GW) relation is equivalent to $\hat{\gamma}_5^2 = 1$. Next define the following complete set of states:

$$\hat{\gamma}_5 u_i = -u_i \;, \qquad \hat{\gamma}_5 w_i = w_i \tag{2.2}$$

$$\hat{P}_- = \sum_i u_i u_i^\dagger \;, \quad \hat{P}_+ = \sum_i w_i w_i^\dagger = 1 - \hat{P}_- \,, \tag{2.3}$$

where we treat u, w as columns and $u^\dagger, w^\dagger$ as rows. For a topologically trivial background, the number of u and w eigenvectors is the same, equal to N^2 each for a two-dimensional square lattice ($2N^2$ total dimension).[5] We also define the eigenvectors of γ_5, which can be chosen independent of the gauge background:

$$\gamma_5 v_i = v_i \;, \qquad \gamma_5 t_i = -t_i \tag{2.4}$$

$$P_+ = \sum_i v_i v_i^\dagger \;, \quad P_- = \sum_i t_i t_i^\dagger = 1 - P_+ \;. \tag{2.5}$$

2.2 Chiral variables, Jacobians, and their variations

Consider a vectorlike lattice theory with partition function:

$$Z_V = \int \prod_x \mathrm{d}\Psi_x \mathrm{d}\bar{\Psi}_x \, e^S \,, \tag{2.6}$$

where x denotes both spinor and spacetime lattice indices. For the time being, we will take the action S to be the usual kinetic action $S = \sum_{x,y} \bar{\psi}_x D_{x,y} \psi_y \equiv (\bar{\Psi} \cdot D \cdot \Psi)$, which has an exact chiral symmetry, $\Psi \to e^{i\alpha\hat{\gamma}_5}\Psi$, $\bar{\Psi} \to \bar{\Psi} e^{i\alpha\gamma_5}$.

Now we change variables from Ψ_x, $\bar{\Psi}_x$ to $c_i^\pm$, $\bar{c}_i^\pm$ defined by the following expansions in terms of the γ_5 and $\hat{\gamma}_5$ eigenvectors (we let x also include spinor index, thus x takes $2N^2$ values in 2d):

$$\Psi_x = \sum_i c_i^+ w_i(x) + c_i^- u_i(x) \tag{2.7}$$

$$\bar{\Psi}_x = \sum_i \bar{c}_i^+ t_i^\dagger(x) + \bar{c}_i^- v_i^\dagger(x) \;. \tag{2.8}$$

[5] Most of the formulae in this paper are valid in any even dimension; in a few obvious instances, however, we specialize to two dimensions. Also, when necessary, we specialize to the case of a $\mathrm{U}(1)$ gauge group.

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The change of variables leads to a Jacobian:

$$\prod_x \mathrm{d}\Psi_x \mathrm{d}\bar{\Psi}_x = \frac{1}{J} \prod_i \mathrm{d}c_i^+ \mathrm{d}c_i^- \mathrm{d}\bar{c}_i^+ \mathrm{d}\bar{c}_i^- \tag{2.9}$$

$$J = \det||w_i(x) u_j(x)|| \det||v_i^\dagger(x) t_j^\dagger(x)|| \,, \tag{2.10}$$

(note that $||w_i(x)u_j(x)||$ is a $2N^2 \times 2N^2$ dimensional matrix, with x indexing rows and i, j-columns) and the partition function becomes:

$$\begin{aligned} Z_V &= \int \prod_x \mathrm{d}\Psi_x \mathrm{d}\bar{\Psi}_x e^S = \frac{1}{J} \int \prod_i \mathrm{d}c_i^+ \mathrm{d}c_i^- \mathrm{d}\bar{c}_i^+ \mathrm{d}\bar{c}_i^- e^{\sum_{i,j} \bar{c}_i^+ c_j^+ (t_i^\dagger \cdot D \cdot w_j) + \bar{c}_i^- c_j^- (v_i^\dagger \cdot D \cdot u_j)} \\ &= \frac{1}{J} \det||(t_i^\dagger \cdot D \cdot w_j)|| \det||(v_i^\dagger \cdot D \cdot u_j)|| \ . \end{aligned} \tag{2.11}$$

Under infinitesimal changes of the gauge field background:

$$U_{x,\mu} \to U_{x,\mu} + \delta_{\eta_{x,\mu}} U_{x,\mu} \ , \tag{2.12}$$

which, in the case of gauge transformations, take the form:

$$\delta_\omega U_{x,\mu}\big|_{\text{gauge}} = i\left(\omega_x U_{x,\mu} - U_{x,\mu}\omega_{x+\mu}\right) \equiv -i\left(\nabla_\mu \omega_x\right) \ U_{x,\mu} \,, \tag{2.13}$$

the various factors in Z_V change as described below.

(i) The change of the "positive chirality" determinant is:

$$\begin{aligned} \delta_\eta \ln \det ||(t_i^\dagger \cdot D \cdot w_j)|| &= \sum_{j,k} (w_j^\dagger \cdot D^{-1} \cdot t_k)(t_k^\dagger \cdot \delta_\eta D \cdot w_j) \\ &\qquad\qquad + (w_j^\dagger \cdot D^{-1} \cdot t_k)(t_k^\dagger \cdot D \cdot \delta_\eta w_j) \\ &= \mathrm{tr}(\hat{P}_+ D^{-1} \delta_\eta D) + \sum_j (w_j^\dagger \cdot \delta_\eta w_j) \ . \end{aligned}$$

To obtain (2.14), in the first line we used $\sum_k (w_j^\dagger \cdot D^{-1} \cdot t_k)(t_k^\dagger \cdot D \cdot w_i) = \delta_{ji}$, while in the second line we used the freedom to insert $\sum_k v_k v_k^\dagger$ (which, using $\hat{P}_+ D^{-1} = D^{-1} P_-$, is killed by the projectors); finally, we used completeness, $\sum_k t_k t_k^\dagger + v_k v_k^\dagger = 1$. The trace in (2.14) is over spinor as well as space-time indices.

We note that the first term in (2.14) reflects the change of the operator, D, while the second is due to the change of basis vectors w_i, which depend on the gauge background (while the t, v-vectors do not). We stress that this factorization of the change of the "positive chirality" determinant into separate terms, one due to the change of the operators and the other due to the change of basis vectors, is a general feature of chiral partition functions. This will be proven for partition functions defined with a general chiral action in section 4.2, and will be important in what follows.

(ii) For the "negative chirality" determinant, using $\sum_k (u_j^\dagger \cdot D^{-1} \cdot v_k)(v_k^\dagger \cdot D \cdot u_i) = \delta_{ji}$, similar to the derivation of (2.14), we find:

$$\begin{aligned}\delta_\eta \ln \det ||(v_i^\dagger \cdot D \cdot u_j)|| &= \sum_{j,k} (u_j^\dagger \cdot D^{-1} \cdot v_k)(v_k^\dagger \cdot \delta_\eta D \cdot u_j) \\ &\qquad\qquad + (u_j^\dagger \cdot D^{-1} \cdot v_k)(v_k^\dagger \cdot D \cdot \delta_\eta u_j), \\ &= \mathrm{tr}(\hat{P}_- D^{-1} \delta_\eta D) + \sum_j (u_j^\dagger \cdot \delta_\eta u_j) \ .\end{aligned}$$

Here, we also have a contribution from the change of operator, the first term in (2.14) as well as a contribution due to the change of basis.

1. Finally, the change of Jacobian is computed from the change of its first factor:

$$\begin{aligned}\delta_\eta \ln \det ||w_i(x) u_j(x)|| &= \sum_{x,y,i,j} \left\| \begin{matrix} w_i^\dagger(x) \\ u_j^\dagger(y) \end{matrix} \right\| \times ||\delta_\eta w_i(x) \delta_\eta u_j(y)|| \\ &= \sum_i (w_i^\dagger \cdot \delta_\eta w_i) + (u_i^\dagger \cdot \delta_\eta u_i) \,, \end{aligned} \tag{2.14}$$

leading to:

$$\frac{1}{J} \to \frac{1}{J} \, e^{-\sum_i \left[(w_i^\dagger \cdot \delta_\eta w_i) + (u_i^\dagger \cdot \delta_\eta u_i) \right]} \ . \tag{2.15}$$

Now we can collect all factors, and find that in the vectorlike theory the factors in (2.14), (2.14), (2.15) having to do with the choice of basis vectors cancel out from the change of the partition function and we are left with:

$$\begin{aligned} Z_V[U + \delta_\eta U] &= Z_V[U] e^{\mathrm{tr}(\hat{P}_+ D^{-1} \delta_\eta D) + \mathrm{tr}(\hat{P}_- D^{-1} \delta_\eta D)} \\ &= Z_V[U] e^{\mathrm{tr} D^{-1} \delta_\eta D} \,, \end{aligned} \tag{2.16}$$

showing that the change of the partition function is determined solely by the change of the GW operator. In particular, for a gauge variation of U, eq. (2.13), we find immediately from (2.16) that $Z_V[U + \delta_\omega U] = Z_V[U]$, and also that:

$$\begin{aligned} \mathrm{tr} \hat{P}_+ D^{-1} \delta_\omega D &= i \mathrm{tr}(P_- - \hat{P}_+)\omega = -\frac{i}{2} \, \mathrm{tr} \omega \hat{\gamma}_5 = -\frac{i}{2} \sum_x \omega_x \, \mathrm{tr}(\hat{\gamma}_5)_{xx} \,, \\ \mathrm{tr} \hat{P}_- D^{-1} \delta_\omega D &= i \mathrm{tr}(P_+ - \hat{P}_-)\omega = \frac{i}{2} \, \mathrm{tr} \omega \hat{\gamma}_5 = \frac{i}{2} \sum_x \omega_x \, \mathrm{tr}(\hat{\gamma}_5)_{xx} \,, \end{aligned} \tag{2.17}$$

where the trace in the last line is over spinor indices only. The field $\mathrm{tr}(\hat{\gamma}_5)_{xx}$ appearing in the basis-independent gauge variations (2.17), is known to be a topological lattice field, which expresses the chiral anomaly on the lattice (this follows, e.g., from the index theorem of [22], see also [13, 47, 23]). Naturally, eqs. (2.17) show that the anomalies due to the left- and right-moving fermions cancel.

Since we will be interested in splitting vectorlike theories' lattice partition functions with more general actions into chiral components, and in the dependence of these chiral components on the gauge field background, we will focus our discussion on the term

$\sum_i (w_i^\dagger \cdot \delta_\eta w_i)$ (or similarly $\sum_i u_i^\dagger \cdot \delta_\eta u_i$) in the following sections and pay great attention to the variations of the basis vectors with respect to changes of the gauge background. Following [13], we will refer to $\sum_i (w_i^\dagger \cdot \delta_\eta w_i)$, and similar for $w_i \to u_i$, as "measure terms" since not only they depend on but also uniquely determine the fermion measure [23].

3. The "1-0" GW-Yukawa model and a paradox

The Yukawa-Higgs-GW-fermion model being considered here, which we call the "1-0" model, is a U(1) two-dimensional lattice gauge theory with one charged Dirac fermion ψ of charge 1 and a neutral spectator Dirac fermion χ.

Considering this theory is motivated by its simplicity: it is the minimal Higgs-Yukawa-GW-fermion model in two dimensions which holds the promise to yield, at strong Yukawa coupling, a chiral spectrum of charged fermions and is, at the same time, amenable to numerical simulations not requiring the use of extensive computing resources. The fermion part of the action of the "1-0" model is:

$$\begin{aligned} S &= S_{\text{light}} + S_{\text{mirror}} \qquad (3.1)\\ S_{\text{light}} &= \left(\bar{\psi}_+ \cdot D_1 \cdot \psi_+\right) + \left(\bar{\chi}_- \cdot D_0 \cdot \chi_-\right)\\ S_{\text{mirror}} &= \left(\bar{\psi}_- \cdot D_1 \cdot \psi_-\right) + \left(\bar{\chi}_+ \cdot D_0 \cdot \chi_+\right)\\ &\quad + y\left\{\left(\bar{\psi}_- \cdot \phi^* \cdot \chi_+\right) + \left(\bar{\chi}_+ \cdot \phi \cdot \psi_-\right) + h\left[\left(\psi_-^T \cdot \phi\gamma_2 \cdot \chi_+\right) - \left(\bar{\chi}_+ \cdot \gamma_2 \cdot \phi^* \cdot \bar{\psi}_-^T\right)\right]\right\} . \end{aligned}$$

The chirality components for the charged and neutral fermions are defined, by projectors including the appropriate Neuberger-Dirac operators (charged D_1 and neutral D_0) for the unbarred components, i.e. $\psi_\pm = (1 \pm \hat{\gamma}_5)\psi/2$. The field $\phi_x = e^{i\eta_x}$, $|\eta_x| \leq \pi$, is a unitary higgs field of unit charge with the usual kinetic term:

$$S_\kappa = \frac{\kappa}{2} \sum_x \sum_{\hat{\mu}} \left[2 - \left(\ \phi_x^* \, U_{x,x+\hat{\mu}} \, \phi_{x+\hat{\mu}} + \text{h.c.}\ \right)\right] . \qquad (3.2)$$

The inclusion of both Majorana and Dirac gauge invariant Yukawa terms is necessitated by the requirement that all global symmetries not present in the desired target chiral gauge theory be explicitly broken, see [42, 12]. Moreover, consistent with the symmetries, if the Majorana coupling h vanishes, there are exact mirror-fermion zero modes for arbitrary backgrounds ϕ_x, which can not be lifted in the disordered phase [41].

From now on, we will call the fermion fields that participate in the Yukawa interactions the "mirror" fields — these are the negative chirality component, ψ_-, of the charged ψ, and the positive chirality component, χ_+, of the neutral χ, while the fields ψ_+ and χ_- will be termed "light."

The lattice action (3.1) completely defines the theory via a path integral over the charged and neutral fermion fields, the unitary higgs field, as well as the gauge fields. We will not consider the integral over the lattice gauge fields, but will study in detail the variation of the partition function with respect to the gauge background.

Our interest is in the symmetric phase of the unitary higgs theory (expected to occur at $\kappa < \kappa_c \simeq 1$), where the higgs field acts — modulo correlations induced by $\kappa \neq 0$ and by fermion backreaction — essentially as a random variable. Based on experience with strong-Yukawa expansions in theories with naive or Wilson fermions, it is expected that in the large-y, fixed-h limit, there is a symmetric phase where the fermions ψ_- and χ_+ decouple from the long distance physics. In the symmetric phase, this decoupling occurs without breaking the chiral symmetry, essentially by forming chiral-neutral composites of the fermions with the scalar ϕ, as described around eq. (1.3) of section 1.2.

The expected spectrum of light fields in the target theory consists of the charged ψ_+ and the neutral χ_-. The spectrum of the mirror theory was investigated numerically in [41], for vanishing gauge field background and in the infinite-y limit. The evidence found there points towards decoupling of the mirror sector, with no breaking of the chiral symmetry of the mirror sector (this symmetry is gauged by the U(1) gauge field). The analysis of ref. [41] was performed by first using the analogue of the formulae from sections 2.1, 2.2, for the case of vanishing gauge background. The eigenvectors of $\hat{\gamma}_5$ were explicitly worked out and the splitting of the partition function into "light" and "mirror" was made manifest. Subsequently, a Monte Carlo simulation of the mirror sector in the infinite-y limit was performed, yielding the above-cited results about the decoupling of all mirror sector fields (decoupling at infinite y further requires $h > 1$). However, a complete decoupling of the mirror is subtle. As discussed in the Addendum, we suspect that light degrees of freedom might still exist in a very contrived way, and further studies are required to clarify this dynamical issue; needless to say, this is under current investigation.

As we showed in section 2.2, the lattice fermion action (3.1) and the corresponding partition function easily split into light and mirror parts also in an arbitrary fixed gauge background. Only the charged eigenvectors (of both light and mirror fields) depend on the background. By analogy with (2.11) we have a split of the partition function:

$$Z[U;y,h] = Z_L[U] \times \frac{1}{J[U]} \times Z_M[U;y,h] \ . \tag{3.3}$$

Here $Z_L[U] = \det ||(t_i^\dagger \cdot D \cdot w_j)|| \times$(determinant of neutral light spectator) is the light sector partition function. The jacobian J is the product of the jacobians (2.9) for the charged and neutral sectors. Finally, Z_M denotes the mirror partition function — an integral over the charged mirrors, neutral mirrors, and unitary higgs field. The mirror fermion integral is a determinant which includes a kinetic term, as in (2.11), but now also the Yukawa terms from (3.1), and is also averaged over the random ϕ_x (we take $\kappa \to 0$).

Now, because the l.h.s. of (3.3) is manifestly gauge invariant, so is the r.h.s., since it is obtained from the l.h.s. simply via a change of variables. But we know how two of the factors on the r.h.s. transform under gauge transformations: the light partition function $Z_L[U]$ and the Jacobian $1/J[U]$, for which we have, from (2.14), using (2.17):

$$\frac{Z_L[U^\omega]}{J[U^\omega]} \simeq \frac{Z_L[U]}{J[U]} \exp\left(-\frac{i}{2}\mathrm{tr}\omega\hat{\gamma}_5 - \sum_i (u_i^\dagger \cdot \delta_\omega u_i)\right) . \tag{3.4}$$

Therefore, from (3.3) and the fact that the l.h.s. is gauge invariant, it follows that the mirror partition function transforms, under gauge transformations, as follows:

$$Z_M[U^\omega; y, h] \simeq Z_M[U; y, h] \exp\left(\frac{i}{2}\mathrm{tr}\omega\hat{\gamma}_5 + \sum_i (u_i^\dagger \cdot \delta_\omega u_i)\right), \tag{3.5}$$

independent not only on the values of the Yukawa couplings (y, h) but also most of the details of the mirror action; we note that our more general considerations of section 4.2 give a direct proof of this result. In passing, we stress that we can not similarly infer the change of $Z_M[U; y, h]$ under arbitrary (i.e., not gauge transformations) changes of background, since we expect that the change of $Z[U; y, h]$ on the l.h.s. of (3.3) under arbitrary variations of U depends on y, h.

The gauge variation of the mirror partition function of eq. (3.5) leads us to a paradox.[6] The exact result (3.5) shows that the gauge transformation of the mirror partition function should precisely cancel that of the light chiral fermion. If the mirror sector only involves heavy degrees of freedom, as the numerical results of [41] suggest, and if these zero-background results persist for arbitrarily small gauge backgrounds (as one is inclined to expect), then the mirror partition function should be a local functional of the gauge background. By (3.5), this local functional's gauge variation must precisely cancel the anomaly of the light chiral fermion. However, this is known to be impossible, as the anomaly is not the variation of a local functional.

In what follows we will argue that this paradox has a natural resolution in the case when dynamical gauge fields are turned on, which can be found using the results of [6] and [13]. We will show that the paradox is (naturally) absent if the anomalies in the light and mirror sectors cancel separately. Moreover, we will argue, in section 5, that the mirror partition function and, more generally, the generating functional for connected correlation functions in the mirror sector, are smooth functions of the gauge field background in the anomaly-free-mirror case only.

4. More on the choice of basis vectors

To explain the resolution mentioned above and the other results alluded to in the last paragraph (to be discussed in section 5), we need to first consider in more detail the variations of the chiral basis vectors under arbitrary changes of the gauge background and the properties of the resulting fermion measure. This is important, since, as we saw in section 2.2 and will show for more general chiral partition functions in section 4.2, the "measure terms" determine the basis-vector-dependent part of the chiral partition functions' variation with the gauge background. They reflect the ambiguity in the phase choice of the chiral partition functions. Interestingly, the "curvature", associated to this term thought of as a connection, is basis independent. Therefore the "measure term" can not be chosen at random. In particular, in the anomalous case (section 4.3) we recall why there is no definition of the "measure terms" which is a smooth function of the gauge field

[6]We thank N. Arkani-Hamed, M. Golterman, B. Holdom, and Y. Shamir for asking pertinent questions about the anomaly.

background. We also explicitly show the singularity of the basis vectors and the associated "measure terms" that were chosen in our analysis of the 1-0 model. In the anomaly-free case (section 4.3.1) we show how to construct a smooth measure in the Wilson-line subspace of gauge field space by cancelling the singularities in the measure precisely with the help of the phase ambiguity.

4.1 Change of basis vectors under arbitrary change of background

The change of chiral partition function under arbitrary changes of the gauge background is of great interest. By eqs. (2.14), (2.14) and also section 4.2, this clearly depends on the change of basis vectors. Hence, we begin this section by studying how the $\hat{\gamma}_5$ eigenvectors change under changes of the gauge background.

In a U(1) gauge theory, a gauge (2.13) and an arbitrary (2.12) change of background differ in the choice of the function η in (2.12):

$$\delta_{\eta_{x,\mu}} U_{x,\mu} \equiv \eta_{x,\mu} U_{x,\mu} \tag{4.1}$$

where for gauge variations we have $\eta_{x,\mu} = -i \nabla_\mu \omega_x$. The $\hat{\gamma}_5$ matrix changes as follows:

$$\hat{\gamma}_5[U + \delta_\eta U] = \hat{\gamma}_5[U] - \gamma_5 \delta_\eta D \,, \tag{4.2}$$

where the second term follows from $\hat{\gamma}_5 = \gamma_5(1 - D)$. The variation $\delta_\eta D$ obeys:

$$\hat{\gamma}_5(\gamma_5 \delta_\eta D) = -(\gamma_5 \delta_\eta D)\hat{\gamma}_5 \,, \tag{4.3}$$

as a consequence of the GW relation (i.e., $\hat{\gamma}_5^2 = 1$).

Now given the set of eigenvectors u_i, w_i of $\hat{\gamma}_5[U]$, obeying orthonormality $(w_i^\dagger \cdot w_j) = (u_i^\dagger \cdot u_j) = \delta_{ij}$ and $(u_i^\dagger \cdot w_j) = 0$, we wish to find the eigenvectors of $\hat{\gamma}_5[U + \delta_\eta U]$ of (4.2):

$$\begin{aligned} (\hat{\gamma}_5 - \gamma_5 \delta_\eta D)\, w_i' &= w_i' \\ (\hat{\gamma}_5 - \gamma_5 \delta_\eta D)\, u_i' &= -u_i' \,. \end{aligned} \tag{4.4}$$

We assume that in the neighborhood of the chosen initial background U the vectors change smoothly under small changes of the gauge background. We thus look for w_i' and u_i' as expansions in terms of the old vectors u_i, w_i:

$$\begin{aligned} w_i' &= w_i + \delta_\eta w_i \,, \quad \delta_\eta w_i = i\alpha_{ij}\, w_j + \beta_{ij}\, u_j \,, \\ u_i' &= u_i + \delta_\eta u_i \,, \quad \delta_\eta u_i = i\gamma_{ij}\, u_j + \kappa_{ij}\, w_j \,, \end{aligned} \tag{4.5}$$

where $\alpha, \beta, \kappa, \gamma$ are assummed linear in η. Substituting (4.5) into the orthonormality relations for the primed vectors, we immediately see that they require that α and γ be hermitean matrices, while $\beta^\dagger = -\kappa$. We now plug (4.5) into (4.4), keeping terms to leading order in δ_η, to find the equations determining the change of the vectors (a sum over repeated indices is assumed):

$$\begin{aligned} (1 - \hat{\gamma}_5)\, \delta_\eta w_i &= -(\gamma_5 \delta_\eta D)\, w^i \;\leftrightarrow\; 2\beta_{ij} u_j = -(\gamma_5 \delta_\eta D) w_i \\ (1 + \hat{\gamma}_5)\, \delta_\eta u_i &= (\gamma_5 \delta_\eta D)\, u^i \;\leftrightarrow\; 2\kappa_{ij} w_j = (\gamma_5 \delta_\eta D) u_i \,, \end{aligned} \tag{4.6}$$

showing that the hermitean matrices α and γ are completely arbitrary, while β and κ are completely determined by $\delta_\eta D$. Finally, for the change of the basis vectors with a change of gauge background, $\delta_\eta w_i$ and $\delta_\eta u_i$ (it is easy to check that $\beta^\dagger = -\kappa$ for the explicit solution below), we find:

$$\begin{aligned} \delta_\eta w_i &= i\alpha_{ij} w_j - \frac{1}{2}\, u_j (u_j^\dagger \cdot \gamma_5 \delta_\eta D \cdot w_i)\,, \\ \delta_\eta u_i &= i\gamma_{ij} u_j + \frac{1}{2}\, w_j (w_j^\dagger \cdot \gamma_5 \delta_\eta D \cdot u_i)\ . \end{aligned} \tag{4.7}$$

Eqs. (4.7) show that the change of the basis vectors with definite chirality in the direction orthogonal to the same chirality subspace is completely determined by the change of the gauge background and that the arbitrariness is in the freedom to make an unitary transformation in the given chirality subspace.

It is also clear from (4.7) that the change of the basis vectors contributing to the change of measure and Jacobian, as in (2.9), can be written, by, e.g., changing the gauge background at a single link only and using linearity of α, γ in η, as follows:

$$\begin{aligned} \sum_i (w_i^\dagger \cdot \delta_\eta w_i) &= i\alpha_{ii} \equiv -\sum_{x,\mu} \eta_{x,\mu}\, j^w_{\mu,x}[U]\,, \\ \sum_i (u_i^\dagger \cdot \delta_\eta u_i) &= i\gamma_{ii} \equiv -\sum_{x,\mu} \eta_{x,\mu}\, j^u_{\mu,x}[U]\ . \end{aligned} \tag{4.8}$$

The "currents" appearing in (4.8) are, generally, functionals of the gauge background as we have indicated above; the left- and right-handed currents j^u and j^w can be different. We stress that the measure terms (4.8) are purely imaginary — it is precisely the U-dependence of the phase of the chiral partition functions that is left ambiguous.

While the perturbative equations (4.4), (4.5) do not determine the currents (4.8), there are important restrictions imposed on them by global considerations [6]. These arise upon considering the second variation of the "measure" terms (4.8), $\delta_\zeta \sum_i (w_i^\dagger \delta_\eta w_i) = \sum_i (\delta_\zeta w_i^\dagger \cdot \delta_\eta w_i) + (w_i^\dagger \cdot \delta_\zeta \delta_\eta w_i)$, in particular the "curvature":[7]

$$f^w_{\zeta\eta} \equiv \sum_i (\delta_\zeta w_i^\dagger \cdot \delta_\eta w_i) - (\delta_\eta w_i^\dagger \cdot \delta_\zeta w_i)\,, \tag{4.9}$$

which can be calculated upon substituting eqs. (4.7) for the variations $\delta_\eta w_i$, $\delta_\zeta w_i$ into (4.9). One notices that $f^w_{\zeta\eta}$ is independent on the undetermined matrices α_{ij} and only depends on the variation of the basis vectors in the orthogonal subspace:

$$\begin{aligned} f^w_{\zeta\eta} &= \frac{1}{4} \sum_{i,j} \Big((w_i^\dagger \cdot \gamma_5 \delta_\zeta D \cdot u_j)(u_j^\dagger \cdot \gamma_5 \delta_\eta D \cdot w_i) - (\zeta \leftrightarrow \eta) \Big) \\ &= \frac{1}{4}\, \mathrm{Tr} \left(\hat{P}_+ \left[\gamma_5 \delta_\zeta D, \gamma_5 \delta_\eta D \right] \right) \\ &= \mathrm{Tr} \left(\hat{P}_+ \left[\delta_\zeta \hat{P}_-, \delta_\eta \hat{P}_- \right] \right)\,, \end{aligned} \tag{4.10}$$

[7] That $f^w_{\zeta\eta}$ is indeed the sum of Berry curvatures for the positive "energy" eigenstates w_i of the "Hamiltonian" $\hat{\gamma}_5$ depending on the parameters $[U]$ is explained in [6].

where we used the relations between eigenvectors and projectors of section 2.1. A similar relation is obtained for the negative chirality curvature:

$$f^u_{\zeta\eta} = \mathrm{Tr}\left(\hat{P}_-\left[\delta_\zeta \hat{P}_+, \delta_\eta \hat{P}_+\right]\right) , \tag{4.11}$$

obeying, of course, $f^u_{\zeta\eta} + f^w_{\zeta\eta} = 0$. The relations (4.10), (4.11) show that the curvatures of the measure terms (4.8) are basis independent and imply that if the curvatures are nonvanishing, the the currents $j^w_\mu[U]$, $j^u_\mu[U]$—depending on the choice of phases (4.7) of the $\hat{\gamma}_5$ eigenvectors w_i, u_i—can not be chosen to be independent on the background (in particular, they can not be taken to vanish).

Most importantly, eqs. (4.10), (4.11) also imply that if perturbative anomalies do not separately cancel among the light and mirror fermions, the measure terms $\sum_i (w_i^\dagger \cdot \delta_\eta w_i)$ can not be chosen to be smooth functions of the gauge field background (see section 4.3 and [6, 13]). This is because the curvature defined above integrates, over some closed sub-manifolds in the gauge field configuration space, to quantized non-zero values.

The "measure" terms determine the variations of the chiral partition functions under changes of the gauge background — as we saw in the example of the vectorlike theory with the usual kinetic action, see (2.11), (2.14), (2.14), (2.15). We will prove this for more general chiral partition functions in section 4.2. The singular nature of the measure implies that the separation of the partition function into chiral "light" and "mirror" components can not be a smooth function of the gauge background if the light and mirror anomalies do not separately cancel and that the separation of the partition function is ill-defined. The explicit manifestation of the singularity in the 1-0 model will be considered in sections 4.3, 4.3.1.

4.2 On the variations of chiral partition functions

Before considering anomaly cancellation and the smoothness of the measure, in this section, we prove an important property on the variation of chiral partition functions. It is a fairly straightforward proof but is also very general. We find it quite useful, and in particular, section 5 contains an example of how such a general proof can lead to some strong conclusions.

Suppose $S[X^a_x,\ Y^{\dagger b}_x,\ O^c_{xy}]$ is an arbitrary action. Here X^a_x and $Y^{\dagger b}_x$ are some d-dimensional ("fundamental" and "anti-fundamental") vectors, a and b are "flavor" indices, and $x = 1, 2, \ldots, d$ labels both spatial and spinor indices. O^c are some additional operators the theory depends on, which we assume to be some $d \times d$ matrices. The action is said to be "chiral" in the following sense. For each flavor X^a and $Y^{\dagger b}$, there exist projection operators $\hat{P}^a$ and P^b respectively, all satisfying $P^2 = P$, such that:

$$S[X^a_x,\ Y^{\dagger b}_x,\ O^c_{xy}] = S[\hat{P}^a_{xy} X^a_y,\ Y^{\dagger b}_y P^b_{yx},\ O^c_{xy}]. \tag{4.12}$$

A summation over all repeated lattice and spinor indices (x, y, z) is understood, here and further in this section. Given any action $\tilde{S}[X, Y^\dagger, O]$ and some projection operators $\hat{P}$ and P, one can always "build" a chiral action by just defining $S[X, Y^\dagger, O] = \tilde{S}[\hat{P}X, Y^\dagger P, O]$. The following property of a "chiral" action is essential for our discussions here. Given any vector u_x such that $\hat{P}^a_{xy} u_y = 0$, one has:

$$\frac{\delta S}{\delta X^a_x}\, u_x = \frac{\delta S}{\delta(\hat{P}^a_{zy} X^a_y)}\, \hat{P}^a_{zx} u_x = 0, \tag{4.13}$$

where equation (4.12) is used in the second step. A similar property holds true for $\delta S/\delta Y^{\dagger b}_x$ as well.

We now proceed to explicitly construct the chiral partition functions by choosing two sets of orthonormal basis w_i and t_j such that:

$$\hat{P} w_i = w_i, \qquad P t_j = t_j \tag{4.14}$$

$$w^\dagger_i w_j = \delta_{ij}, \qquad t^\dagger_i t_j = \delta_{ij}\ . \tag{4.15}$$

Using chirality, we then set $X = \sum_i c_i w_i$, $Y^\dagger = \sum_i \bar{c}_i t^\dagger_i$ and define the partition function, specializing without loss of generality to the case of one flavor:

$$Z[U] = \int \prod_i \mathrm{d}c_i \prod_j \bar{\mathrm{d}}c_j\ e^{S\left[\sum\limits_i c_i w_i,\, \sum\limits_j \bar{c}_j t^\dagger_j,\, O\right]}. \tag{4.16}$$

We also note that no assumptions about the locality, (bi-)linearity, etc., of S are made here; in particular, S may be an effective action for the chiral fermions X and Y obtained after integration over some other degrees of freedom; its chirality (4.12) is the only important property for what follows. For example, S could be the mirror-fermion effective action obtained after integrating over the random ($\kappa \to 0$) unitary higgs field in the "1-0" model with mirror action given in (3.1), (3.2).

We now imagine that the projectors $\hat{P}$ as well as the operator(s) O depend on some external fields U, inducing external field dependence of $Z[U]$ as indicated in (4.16); here we will assume that the projector P can also depend on U, although in our application this will not be the case. We wish to compute the variation of $Z[U]$ under changes of the background field:

$$w_i \to w_i + \delta w_i, \qquad t_i \to t_i + \delta t_i, \tag{4.17}$$

and

$$O \to O + \delta O\ . \tag{4.18}$$

The variation of S is given by:

$$\delta S = \frac{\delta S}{\delta X_x} \sum_i c_i\, \delta w_{ix} + \sum_j \bar{c}_j\, \delta t^\dagger_{ix}\, \frac{\delta S}{\delta Y^\dagger_x} + \frac{\delta S}{\delta O}\, \delta O\ . \tag{4.19}$$

The variation of the partition function is, therefore:

$$\begin{aligned} \delta Z[U] &= \int \prod_i \mathrm{d}c_i \prod_j \mathrm{d}\bar{c}_j\ e^S\ \delta S \\ &= Z[U] \cdot \left[\sum_i \left\langle \frac{\delta S}{\delta X_x} c_i \right\rangle \delta w_{ix} + \sum_j \delta t^\dagger_{jx} \left\langle \bar{c}_j \frac{\delta S}{\delta Y^\dagger_x} \right\rangle + \left\langle \frac{\delta S}{\delta O}\, \delta O \right\rangle \right]. \end{aligned} \tag{4.20}$$

Here and below, “$< >$” denotes expectation values.

Now, the following identity of an arbitrary grassmann integral is easily verified

$$\int \prod_i \mathrm{d}c_i \, \frac{\delta F(c_1, c_2, \dots)}{\delta c_k} \, c_l = \delta_{kl} \int \prod_i \mathrm{d}c_i \; F(c_1, c_2, \dots). \tag{4.21}$$

Here F is an arbitrary function of multiple grassmann numbers. We are being very casual with the ordering of grassmann numbers — this identity holds only if the ordering of the grassmann numbers are defined such that it's unchanged before and after the variation on F. We will implicitly assume this rule in the following calculations.

It is amusing that $\left\langle \frac{\delta S}{\delta X_x} \, c_i \right\rangle$ and $\left\langle \bar{c}_j \, \frac{\delta S}{\delta Y_x} \right\rangle$ can be computed without knowing the actual form of S at all, essentially as a direct consequence of identity (4.21) and the chirality of the action. We claim that:

$$\left\langle \frac{\delta S}{\delta X_x} \, c_i \right\rangle = w^\dagger_{ix} \qquad \text{and} \qquad \left\langle \bar{c}_j \, \frac{\delta S}{\delta Y_x} \right\rangle = t_{jx}. \tag{4.22}$$

To prove (4.22), one only needs to verify the inner products as:

$$\begin{aligned} \left\langle \frac{\delta S}{\delta X_x} \, c_i \right\rangle w_{jx} &= \frac{1}{Z} \int \prod_k \mathrm{d}c_k \prod_l \mathrm{d}\bar{c}_l \;\; e^S \, \frac{\delta S}{\delta X_x} \, c_i \, w_{jx} \\ &= \frac{1}{Z} \int \prod_k \mathrm{d}c_k \prod_l \mathrm{d}\bar{c}_l \; \frac{\delta e^S}{\delta c_j} \, c_i = \delta_{ij} \end{aligned} \tag{4.23}$$

with the help of identity (4.21) in the last step. For any other vector u_x that is perpendicular to all the w_i's one has:

$$\left\langle \frac{\delta S}{\delta X_x} c_i \right\rangle \; u_x = \frac{1}{Z} \int \prod_k \mathrm{d}c_k \prod_l \mathrm{d}\bar{c}_l \; e^S \, \frac{\delta S}{\delta X_x} \, u_x \, c_i = 0 \tag{4.24}$$

simply because $\frac{\delta S}{\delta X_x} \, u_x = 0$, following from chirality of the action, eq. (4.13). Similar properties are easily verified for $\left\langle \bar{c}_i \frac{\delta S}{\delta Y^\dagger_x} \right\rangle$. Since the eigenvectors of $\hat{P}$ and the ones orthogonal to them form a complete set, these conditions are enough to conclude that equation (4.22) holds true.

Therefore the variation (4.20) of the partition function (4.16) becomes, using (4.22):

$$\delta \log Z[U] = \sum_i (w^\dagger_i \cdot \delta w_i) + \sum_i (\delta t^\dagger_i \cdot t_i) + \left\langle \frac{\delta S}{\delta O} \, \delta O \right\rangle \; . \tag{4.25}$$

We thus showed that the factorization property of the variations of chiral actions alluded to after eq. (2.14) is general — the variation of a chiral partition function always factorizes into a variation of the basis vectors plus a variation of the operators.

In the particular case when $\delta t_i = 0$, $P = (1 - \gamma_5)/2$, $\hat{P} = (1 + \hat{\gamma}_5)/2$, and $Z[U]$ is the partition function of, say, the positive chirality fermion — defined by keeping the $c^+, \bar{c}^+$ integral in (2.11) only and equal to det $(t^\dagger_i \cdot D \cdot w_j)$—it is clear that its variation, eq. (2.14), is reproduced by (4.25).

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This theorem is also useful for determining how the chiral partition function transforms under any symmetry the original action $S[X,Y^\dagger,O]$ happens to possess. For example, suppose the action respects the gauge symmetry, namely:

$$0=\delta_\omega S=\frac{\delta S}{\delta X}\delta_\omega X+\delta_\omega Y^\dagger\frac{\delta S}{\delta Y^\dagger}+\frac{\delta S}{\delta O}\delta_\omega O, \tag{4.26}$$

where:

$$\delta_\omega X=i\omega X,\qquad \delta_\omega Y=i\omega Y,\qquad \text{and}\quad \delta_\omega O=i[\omega,\,O\,], \tag{4.27}$$

is the usual gauge transformation on the lattice. Choosing $P=(1+\gamma_5)/2$ and $\hat{P}=(1-\hat{\gamma}_5)/2$ and switching the notation $w\to u$ and $t\to v$ for consistency, equations (4.25) together with (4.26) immediately imply that under this transformation:

$$\begin{aligned}\delta_\omega\log Z&=\sum_i(u_i^\dagger\cdot(\delta_\omega u_i-i\omega u_i))+i\sum_i(v^\dagger\cdot\omega\,v)\\&=\sum_i(u_i^\dagger\cdot\delta_\omega u_i)-i\mathrm{Tr}\,\omega(\hat{P}-P)=\sum_i(u_i^\dagger\cdot\delta_\omega u_i)+\frac{i}{2}\mathrm{Tr}\,\omega\hat{\gamma}_5\end{aligned} \tag{4.28}$$

in agreement with equation (3.5). This procedure applies to more general situations. We will make further use of eq. (4.25) in the following sections.

4.3 Anomaly cancellation and smoothness: the Wilson line background

We now return to the issue of anomaly cancellation and the smoothness of the light-mirror split of the partition function. It is well known that the existence of gauge anomaly in chiral theories is deeply connected to the topology of the gauge field configuration space (for a discussion in the continuum, see [43], while for lattice overlap work, see [6, 23, 44]). On the 2-d square lattice with U(1) gauge group, in a given topological (flux) sector of admissible fields, this space is a N^2+2 dimensional torus times a contractible space [13] and the gauge anomaly prevents one from defining a smooth fermion measure in the path-integral over this space. The general properties of the gauge anomaly are discussed in [13, 23], where it is proven that so long as the anomaly cancellation condition is satisfied a smooth measure exists.

In this section, following [6], we focus on two dimensional chiral theories with only homogeneous Wilson lines turned on, excluding all other gauge field configurations (in sufficiently small volume, the Wilson lines give the leading contribution to the gauge path integral). The use of such a simplified background is that it allows us to explicitly construct the fermion measure and literally see where the singularities appear and how anomaly cancellation removes the difficulty.

The two-dimensional theory is defined on a $N\times N$ lattice. All the fields are endowed with periodic boundary conditions. The gauge field configuration space in this sub-theory is completely tractable. We take the Wilson lines, denoted as $\mathbf{h}=(h_1,h_2)$, to be valued in the range $[0\,,2\pi)$. Physical quantities depending on them must be periodic functions with period 2π. This is the remnant of the general gauge symmetry in this sub-theory. As a result, the variable $\mathbf{h}$ is valued on a two-torus defined by identifying the opposite sides of the square $[0\,,2\pi]\times[0\,,2\pi]$. We shall refer to this torus as the h-torus, or T^2_h in the following.

We would like to demonstrate how anomaly cancellation leads to a smooth measure in such a simplified example. First, we recall some known results of importance. Consider the theory defined by a chiral action $S[X, Y^\dagger, O]$ that satisfies the chirality property (4.12). We assume that only the "$\wedge$-ed" projectors depend on the Wilson lines. As is generally true in chiral theories on the lattice, the partition function is only defined with sets of basis vectors chosen for each projection operator. Suppose they are chosen as:

$$\hat{P}_-(\mathbf{h})\, u_i(\mathbf{h}) = u_i(\mathbf{h}), \qquad P_+ v_i = v_i\,, \tag{4.29}$$

$$\hat{P}_+(\mathbf{h})\, w_i(\mathbf{h}) = w_i(\mathbf{h}), \qquad P_+ t_i = t_i\,, \tag{4.30}$$

and define the partition function as usual:

$$Z(\mathbf{h}) = \int \prod_i \mathrm{d}c_i^- \mathrm{d}\bar{c}_i^+ \cdot \exp\left(S\left[\sum_i c_i^- u_i, \sum_j \bar{c}_j^+ v_j^\dagger, O \right] \right). \tag{4.31}$$

As we know, $Z(\mathbf{h})$ defined in such a way depends on the choice of the basis. In particular if we had chosen $u_i' = \mathrm{U}(\mathbf{h})_{ij} u_j$, where U is some $\mathbf{h}$ dependent unitary matrix, the partition function defined with the new basis defers from the old one by a pure phase $\det \mathrm{U}(\mathbf{h})$.

As explained in section 4.2, given any chosen basis, the variation of the chiral partition function (4.25) consists of two terms:

$$\delta \log Z = \sum_i (u_i^\dagger \cdot \delta u_i) + \frac{1}{Z} \int \prod_i \mathrm{d}c_i^- \mathrm{d}\bar{c}_i^+\, e^S\, \frac{\delta S}{\delta O}\delta O, \tag{4.32}$$

where only the first term, referred to as the "measure term," depends on the basis choice and it uniquely determines the fermion measure [23]. In what follows we denote it by:

$$\mathcal{J}_\mu = \sum_i (u_i^\dagger \cdot \partial_\mu u_i), \tag{4.33}$$

where $\partial_\mu \equiv \frac{\partial}{\partial h_\mu}$. We will also refer to it as the "connection" [6] because the "curvature" associated to it defined as $f_{\mu\nu} = \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu$ plays an important role in our discussion here. As we have derived in section 4.1, see eqs. (4.9), (4.10), the curvature is given by:

$$f_{\mu\nu} = \sum_i (\partial_\mu u_i^\dagger \cdot \partial_\nu u_i) - (\partial_\nu u_i^\dagger \cdot \partial_\mu u_i) = \mathrm{Tr}\left(\hat{P}_-[\,\partial_\mu \hat{P}_-\,,\, \partial_\nu \hat{P}_-\,] \right), \tag{4.34}$$

and is independent on the basis choice. Furthermore, its integral over the entire T^2_h is not difficult to compute [6]. In the case of a single charge-1 chiral fermion with projector $\hat{P}_- = \frac{1-\hat{\gamma}_5}{2}$ (as will be further discussed below in section 4.3.1) the integral of the curvature over the h-torus turns out to be:

$$\int_{T^2_h} f_{\mu\nu} = -2\pi i\;. \tag{4.35}$$

Eq. (4.35) immediately implies that there does not exist an everywhere smooth "connection" $\mathcal{J}_\mu$ defined on the h-torus since $\partial T_h^2 = \varnothing$. Given any chosen basis of u_i's, $\mathcal{J}_\mu$ must always be singular at least at some isolated points on T_h^2. More generally, with multiple charged fermions, each fermion flavor of charge q contributes to the curvature a term $\pm q^2 f_{\mu\nu}(q\mathbf{h})$, where the sign depends on the chirality. Even if the anomaly free condition is satisfied, namely $\sum q_+^2 = \sum q_-^2$, the total curvature $f_{\mu\nu}^{\mathrm{TOT}} = \sum_{q_-} q_-^2 f_{\mu\nu}(q_-\mathbf{h}) - \sum_{q_+} q_+^2 f_{\mu\nu}(q_+\mathbf{h})$ does not vanish since each term in the summation varies with $\mathbf{h}$ differently. Its integral over T_h^2, however, does vanish:

$$\int_{T_h^2} \sum_{q_\pm} f_{\mu\nu}^{q_\pm} = 2\pi i \left(\sum q_+^2 - \sum q_-^2 \right) = 0. \tag{4.36}$$

It then allows for a smooth "connection:"

$$\mathcal{J}_\mu = \sum_{i,q_-} (u_i^{\dagger\, q_-} \cdot \partial_\mu u_i^{q_-}) + \sum_{i,q_+} (w_i^{\dagger\, q_+} \cdot \partial_\mu w_i^{q_+}), \tag{4.37}$$

to be defined on T_h^2. Recall that the measure term is the basis-dependent variation of the chiral partition function. Hence, if the measure term can be chosen to be smooth, a smooth fermion measure exists, at least in this subspace of the gauge field space; see [13] for a general proof of the existence of smooth measure in anomaly-free $\mathrm{U}(1)$ lattice gauge theories and [23] for arguments in the nonabelian case.

4.3.1 Defining the measure of the anomaly free chiral partition function

We will demonstrate how such a smooth measure can be found in the anomaly free theories by first choosing an explicit set of basis vectors.

Notice that the Wilson lines are a homogeneous background and the theory has a translational symmetry, hence it is convenient to work with the momentum eigenstates. On the lattice of size $N \times N$, momenta are discretized in units of $\frac{\pi}{N}$. With the Wilson lines turned on, the momenta effectively become continuous. The Wilson line background shifts the values of momenta in physical observables that depend on them by an amount of $\frac{\mathbf{h}}{2N}$, as we will see in the following. With $h_{1,2}$ defined to take their values in $[0\,, 2\pi)$, this shift exactly "fills in" the gaps between the discrete momenta. Therefore, momenta shifted by the Wilson lines live on 2-torus defined by identifying the opposite sides of the square $[0\,,\pi] \times [0\,,\pi]$. We will refer to this torus (the Brillouin zone) as the momentum-torus, or T_k^2.

To proceed with the explicit construction and choose a basis, we first define the following functions:

$$a(\mathbf{p}) = 1 - \frac{1 - 2s_1^2 - 2s_2^2}{\sqrt{1 + 8s_1^2 s_2^2}},\ b(\mathbf{p}) = \frac{2s_2c_2}{\sqrt{1 + 8s_1^2 s_2^2}},\ c(\mathbf{p}) = \frac{2s_1c_1}{\sqrt{1 + 8s_1^2 s_2^2}}, \tag{4.38}$$

where $s_{1,2} \equiv \sin p_{1,2}$ and $c_{1,2} \equiv \cos p_{1,2}$. The "momenta" $\mathbf{p} = (p_1, p_2) \in T_k^2$ live on the momentum-torus. The functions $a(\mathbf{p})$, $b(\mathbf{p})$ and $c(\mathbf{p})$ just defined are periodic functions of period π and therefore smooth and well-defined everywhere on T_k^2. In momentum space, the two-dimensional Neuberger-Dirac operator $D = 1 - \gamma_5 \hat{\gamma}_5$, see (2.1), with $\gamma_5 = \sigma_3$, has the form:

$$D(\mathbf{p}) = \begin{pmatrix} a_{\mathbf{p}} & -ic_{\mathbf{p}} - b_{\mathbf{p}} \\ -ic_{\mathbf{p}} + b_{\mathbf{p}} & a_{\mathbf{p}} \end{pmatrix} , \tag{4.39}$$

and the GW relation $\hat{\gamma}_5^2 = 1$ is equivalent to $a_{\mathbf{p}}^2 + b_{\mathbf{p}}^2 + c_{\mathbf{p}}^2 = 2a_{\mathbf{p}}$.

Let us first focus on the case of a single fermion of charge 1. A particularly simple choice of the basis vectors is given below [41]. For $\hat{P}_-$ eigenvectors, we choose:

$$u_{\mathbf{k},\mathbf{h}} = \frac{1}{\sqrt{2}N}\, e^{i2\mathbf{k}\cdot\mathbf{x}} \begin{bmatrix} \sqrt{a(\mathbf{k} + \frac{\mathbf{h}}{2N})} \\ i\sqrt{2 - a(\mathbf{k} + \frac{\mathbf{h}}{2N})}\, e^{i\varphi_{\mathbf{k}+\frac{\mathbf{h}}{2N}}} \end{bmatrix} , \tag{4.40}$$

and for $\hat{P}_+$:

$$w_{\mathbf{k},\mathbf{h}} = \frac{1}{\sqrt{2}N}\, e^{i2\mathbf{k}\cdot\mathbf{x}} \begin{bmatrix} i\sqrt{2 - a(\mathbf{k} + \frac{\mathbf{h}}{2N})} \\ \sqrt{a(\mathbf{k} + \frac{\mathbf{h}}{2N})}\, e^{i\varphi_{\mathbf{k}+\frac{\mathbf{h}}{2N}}} \end{bmatrix} . \tag{4.41}$$

Here we defined the phase factor:

$$e^{i\varphi_{\mathbf{p}}} \equiv \frac{ib_{\mathbf{p}} + c_{\mathbf{p}}}{\sqrt{b_{\mathbf{p}}^2 + c_{\mathbf{p}}^2}}, \tag{4.42}$$

and the momenta:

$$\mathbf{k} = \left(\frac{n\pi}{N}, \frac{m\pi}{N}\right), \qquad n, m = 0, 1, \ldots, N-1. \tag{4.43}$$

For the projectors $P_\pm$ that are independent of the Wilson lines we simply choose:

$$v^\dagger_{\mathbf{k},\mathbf{h}} = \frac{1}{N} e^{-i2\mathbf{k}\cdot\mathbf{x}} (1 \quad 0), \qquad t^\dagger_{\mathbf{k},\mathbf{h}} = \frac{1}{N} e^{-i2\mathbf{k}\cdot\mathbf{x}} (0 \quad 1) . \tag{4.44}$$

In the chiral theory defined by an action $S[X, Y^\dagger, O] = S[\hat{P}_- X, Y^\dagger P_+, O]$, only u's (4.40) and v's (4.44) will be involved. Besides the "wave-function" $e^{i2\mathbf{k}\cdot\mathbf{x}}$ (which can be varied by a gauge transformation), everything just defined indeed depends only on the combination $\mathbf{p} = \mathbf{k} + \frac{\mathbf{h}}{2N}$. We find the following picture sometimes helpful. One can imagine that the discretized momenta $\mathbf{k}$ sit on the sites of a $N \times N$ square lattice on the momentum torus T^2_k. The effect of the Wilson line $\mathbf{h}$ is to shift this lattice around T^2_k. When $\mathbf{h}$ goes one cycle around T^2_h, this lattice is shifted exactly by one unit cell and overlaps with the original.

Notice that the function $e^{i\varphi_{\mathbf{p}}}$ of (4.42) is ill-defined[8] at $\mathbf{p} = \mathbf{k} + \mathbf{h}/(2N) = (0,0), (\frac{\pi}{2}, \frac{\pi}{2}), (0, \frac{\pi}{2})$, and $(\frac{\pi}{2}, 0)$. Given that $\mathbf{k}$ is discretized (4.43), these points are only (for unit values

[8]The zero gauge background vectors used to split the partition function in [41] have a discontinuity at these values of momenta.

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of the charge) reached by certain modes of $\mathbf{k}$ when $\mathbf{h} = (0,0) \mod 2\pi$. As a consequence, the "connection:"

$$\mathcal{J}_\mu(\mathbf{h}) = \sum_{\mathbf{k}} (u^\dagger_{\mathbf{k},\mathbf{h}} \cdot \partial_\mu u_{\mathbf{k},\mathbf{h}}) = \frac{i}{2} \sum_{\mathbf{k}} \left(2 - a\left(\mathbf{k} + \frac{\mathbf{h}}{2N}\right)\right) \partial_\mu \varphi_{\mathbf{k}+\frac{\mathbf{h}}{2N}}\,, \tag{4.45}$$

as a vector field defined on T^2_h, can be singular only at $\mathbf{h} = (0,\,0)$ and is perfectly smooth everywhere else. The singularity at $\mathbf{h} = (0,\,0)$ is expected since we already know from (4.35) as well as [6], that the "curvature" associated to "connection" $\mathcal{J}_\mu$ integrates over the entire torus to $2\pi i$. (The curvature $f_{\mu\nu}$ and its integral can also be explicitly computed from (4.45), using (4.42), (4.38).) Imagine on T^2_h we draw a small circular disc D of radius r centered at $\mathbf{h} = (0,\,0)$; then, by Stoke's theorem:

$$\int_{\partial D} \mathcal{J}_\mu = -\int_{T^2_h - D} f_{\mu\nu}. \tag{4.46}$$

Since $f_{\mu\nu}$ is smooth and finite everywhere on T^2_h, in the limit $r \to 0$,

$$\lim_{r\to 0} \int_{\partial D} \mathcal{J}_\mu = -\int_{T^2_h} f_{\mu\nu} = 2\pi i. \tag{4.47}$$

Together with symmetry considerations, we are led to the conclusion that $\mathcal{J}_\mu$ diverges as:

$$\mathcal{J}_\mu(\mathbf{h} \to 0) \approx \frac{1}{r}\hat{\theta}_\mu. \tag{4.48}$$

Here $r = \sqrt{h_1^2 + h_2^2}$ and $\hat{\theta}_\mu$ denotes the unit vector tangential to ∂D. Exactly this $\frac{1}{r}$-singularity is what prevents us from defining the measure smoothly. If we think of $\mathcal{J}_\mu$ as a vector field defined on T^2_h, this singularity appears as a divergent vortex around the singular point. As explained above, this is the only singularity of $\mathcal{J}_\mu$, and if removed, $\mathcal{J}_\mu$ is smooth.

Let us generalize these results to fermions of charge q in a simple manner. Just replace all the $\mathbf{h}$'s in the expressions for $u_{\mathbf{k},\mathbf{h}}$ and $w_{\mathbf{k},\mathbf{h}}$ by $q\mathbf{h}$. The "measure term" is modified to:

$$\mathcal{J}^q_\mu(\mathbf{h}) = \sum_{\mathbf{k}} (u^\dagger_{\mathbf{k},q\mathbf{h}} \cdot \partial_\mu u_{\mathbf{k},q\cdot\mathbf{h}}) = q\ \mathcal{J}^1_\mu(q\mathbf{h}). \tag{4.49}$$

Near every singular point of $\mathcal{J}^q_\mu$, the properties just discussed above continue to hold. For example, near the point $\mathbf{h} = (0,\,0)$, where the measure term $\mathcal{J}^q_\mu$ diverges for any value of q, we have:

$$\mathcal{J}^q_\mu(\mathbf{h} \to 0) = q\mathcal{J}^1_\mu(q\mathbf{h} \to 0) \approx q \cdot \frac{1}{qr}\hat{\theta} = \frac{1}{r}\hat{\theta}. \tag{4.50}$$

Hence, its line integral around the singularity is still $2\pi i$. However, as $\mathcal{J}^q_\mu$ depends on $\mathbf{h}$ through $q\mathbf{h}$, by the periodic properties of $\mathcal{J}^1_\mu$, the number of locations where $\mathcal{J}^q_\mu$ diverges increases to q^2. Indeed, instead of having singularity only at $\mathbf{h} = (0,\,0)$, the same type of singularity must repeat itself at every point where $\mathbf{h} = (\frac{2n\pi}{q},\, \frac{2m\pi}{q}), n, m = 0, 1, \ldots, q-1$, exactly the right amount to account for the integral of $f^q_{\mu\nu}$ over T^2_h that scales as q^2. Figure 1 illustrates the 16 singularities of $\mathcal{J}^4_\mu$ on T^2_h given by a chiral fermion of charge-4.

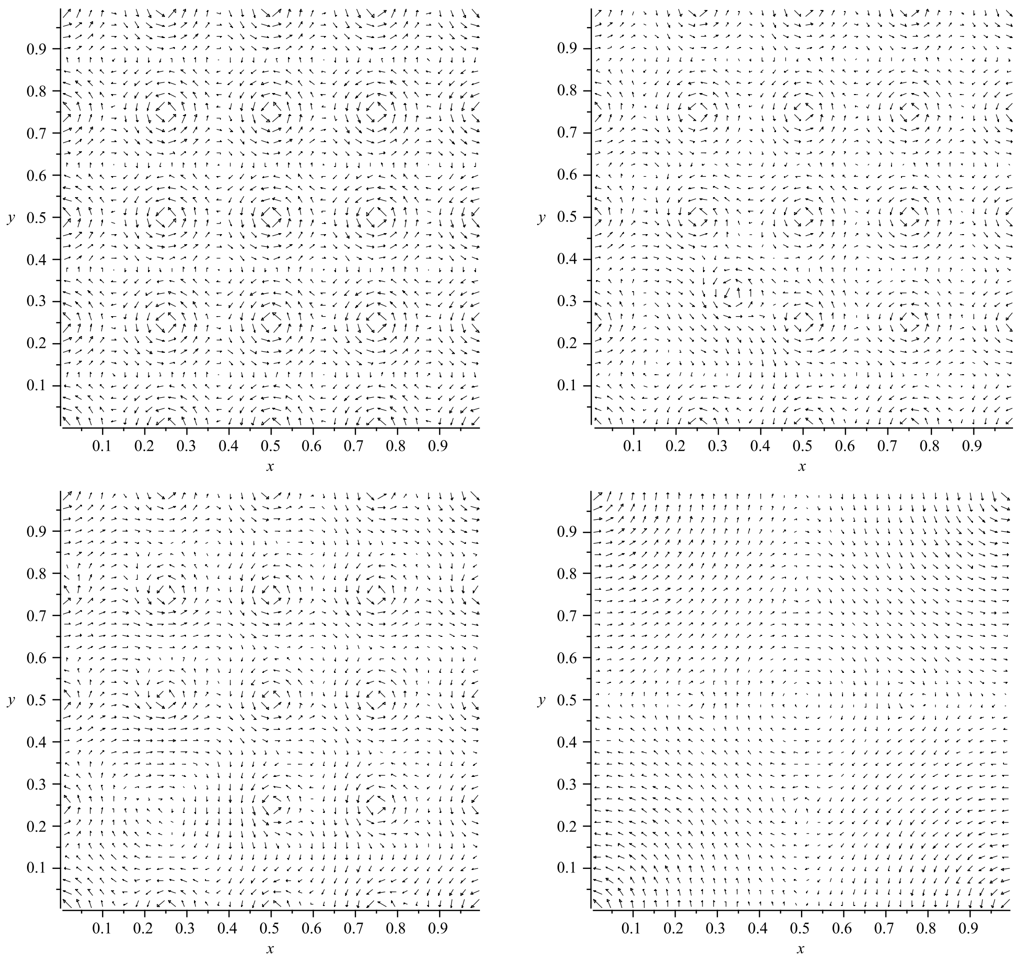

Figure 1: First panel: the 16 singularities of $\mathcal{J}^4_\mu$, each vortex has a divergence at the center. Second: one vortex is slightly shifted. Third: one vortex is moved all the way to $\mathbf{h} = (0, 0)$ so that two singularities coincide there; Fourth: all the vortices are shifted to the corner (the strength of the singularity in the fourth panel is scaled to that of a single vortex). Axes $(x, y) = (h_1/2\pi, h_2/2\pi)$.

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Each vortex indicates a $\frac{1}{r}$-type divergence at its center. The four corners and the two opposite sides are to be identified.

Evidently, the singularities we just discovered are the manifestation of the topological obstruction that prevents one from defining a smooth measure for the anomalous chiral theory [6, 13]. We now focus on the anomaly free case, namely when $\sum q_+^2 = \sum q_-^2$ is satisfied. Obviously fermions with opposite chirality produce vortices with opposite signs, and if they sit on top of each other, they cancel. The anomaly-free condition guarantees that there are always equal number of $+$ and $-$ vortices, giving a nice understanding of the fact that the integral of $f^{\text{total}}_{\mu\nu}$ over the T^2_h vanishes. For the purpose of defining a smooth measure though, this is not sufficient, since normally $+$ and $-$ vortices do not just sit on top of each other. With the current choice of basis, the singularities produced by each charge-q chiral fermion are distributed on T_h^2 with equal separations in both directions. This regularity is very helpful for the counting, but not for the smoothness of the measure. In the "345" model for an example, only the vortices at $\mathbf{h} = (0,\,0)$ overlap and all the rest miss each other. As the consequence, $\mathcal{J}_\mu = \mathcal{J}_\mu^3 + \mathcal{J}_\mu^4 - \mathcal{J}_\mu^5$ diverges at many places.

To find a smooth measure term $\mathcal{J}_\mu$, one only needs to utilize the freedom of basis choice. With the translational symmetry of the Wilson line background to be respected, we are left with only the option to multiply the vectors by some $\mathbf{h}$ dependent phases. This turns out to be sufficient. If we choose to replace the basis vectors (4.40) $u_{\mathbf{k},q\mathbf{h}} \to u_{\mathbf{k},q\mathbf{h}} e^{i\sigma_{\mathbf{k},q\mathbf{h}}}$, the new measure reads:

$$\mathcal{J}_\mu \to \mathcal{J}_\mu + i \sum_{\mathbf{k},q} \partial_\mu \sigma_{\mathbf{k},q\mathbf{h}}. \tag{4.51}$$

The additional term $\partial_\mu \left(\sum \sigma\right)$ is a total derivative of some function defined on the h-torus. If it has no singularities of its own, it certainly does nothing interesting. If it has the same vortex-type singularities as those found in the measure, positive and negative vortices must appear in pairs, because the curl of $\partial_\mu \sigma$ vanishes. Therefore, one can imagine designing such a σ_q, so that $\partial_\mu \sigma_q$ has at least a pair of $+$ and $-$ divergent vortices and one of the them coincides with one of the singularities of $\mathcal{J}_\mu^q$ but with an opposite sign. This will cancel that particular singularity of $\mathcal{J}_\mu^q$ at the position where it was, but will create it elsewhere. The net effect is that singularities can be moved at will through such a manipulation. The second panel of figure 1 demonstrates a particular choice of σ that slightly shifts one of the singularities emerging in $\mathcal{J}_\mu^4$.

We can now envision how a smooth measure can be defined in the anomaly-free case. Simply design the function σ_q such that all the singularities of $\mathcal{J}_\mu^q$ are shifted to a common place so that they can be cancelled by the singularities of appropriate opposite-chirality fermions. A simple way of doing so is to move every vortex toward $\mathbf{h} = (0,\,0)$. During the procedure, one might wish to preserve the lattice rotational symmetry. Such a constraint can be obeyed by moving the singularities in a Z_4 symmetrical way, as pictorially illustrated by figure 2 for moving the singularities of $\mathcal{J}_\mu^3$ and $\mathcal{J}_\mu^2$ respectively.

An explicit expression for σ that realizes the manipulations illustrated in figure 2 can be constructed by first defining $T(x) = \tan\left(\frac{x-\pi}{2}\right)$, and for the charge-2 term $\mathcal{J}_\mu^2$ as an example, choose σ to be ($e^{i\sigma}$ is to be applied on only one of the basis vectors $u_{\mathbf{k},q\mathbf{h}}$):

$$\begin{aligned}
\sigma(h_1,h_2) = \frac{1}{4}\Bigg[&\tan^{-1}\frac{T(h_2)}{T(h_1-\pi)-T(h_1)} - \tan^{-1}\frac{T(2\pi-h_2)}{T(h_1-\pi)-T(h_1)} \\
&- \tan^{-1}\frac{T(h_2)}{T(\pi-h_1)-T(2\pi-h_1)} + \tan^{-1}\frac{T(2\pi-h_2)}{T(\pi-h_1)-T(2\pi-h_1)}\Bigg] \\
+\frac{1}{4}\Bigg[&-\tan^{-1}\frac{T(h_1)}{T(h_2-\pi)-T(h_2)} + \tan^{-1}\frac{T(h_1)}{T(\pi-h_2)-T(2\pi-h_2)} \\
&+\tan^{-1}\frac{T(2\pi-h_1)}{T(h_2-\pi)-T(h_2)} - \tan^{-1}\frac{T(2\pi-h_1)}{T(\pi-h_2)-T(2\pi-h_2)}\Bigg] \\
-\frac{1}{2}&\tan^{-1}\frac{T(h_2)}{T(h_1)} + \frac{1}{2}\tan^{-1}\frac{T(h_1)}{T(h_2)}\,.
\end{aligned} \tag{4.52}$$

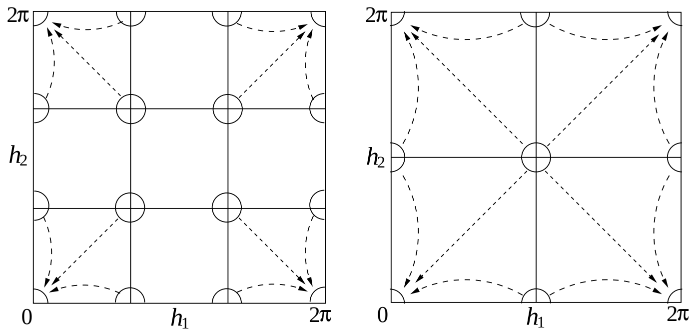

Figure 2: A pictorial illustration for moving the singularities of $\mathcal{J}^3_\mu$ and $\mathcal{J}^2_\mu$ respectively. Each circle represents a divergent vortex and the arrows denote how they should be shifted. The second panel represents the operation described by eq. (4.52).

As long as all the $\frac{1}{r}$-type singularities are cancelled, the new measure $\mathcal{J}_\mu + i\partial_\mu \left(\sum \sigma\right)$ is smooth everywhere.

5. Resolution of the paradox and smoothness of the mirror partition function in anomaly-free case

The resolution of the paradox of section 3 should be evident by now. The basis vectors used to split the light and mirror partition functions in the "1-0" model have a discontinuity at the four special points in momentum space (see footnote before eq. (4.45)). This discontinuity of the basis vectors causes them and the measure to be singular already when only the Wilson line sector of gauge field space is considered (at $h_\mu = 0$). This singularity can not be removed by redefining their phases and is related to the nonvanishing chiral anomaly in the mirror and light sectors. Thus, while the results of [41] hold at $U = 1$, the mirror partition function, generating functional, and spectrum are not smooth functions of the gauge background, and the trivial background results can not be used to infer anything about the spectrum when gauge background fluctuations are included.

More interestingly than resolving the paradox, however, our results from section 4.2, combined with those of ref. [13] (proving that the smooth measure exists iff the anomalies cancel) imply that the mirror generating functional of a Yukawa-Higgs-GW model will be a smooth function of the gauge background whenever the mirror and light sectors are separately anomaly free (the proof below holds in finite volume). Most generally, we wish to prove that, given the "measure term":

$$\mathcal{J}_\mu = \sum_i (u_i^\dagger \cdot \delta_\mu u_i)\,, \tag{5.1}$$

is smooth (here δ_μ indicates variations in all possible directions in gauge field configuration space) the partition function defined by

$$Z = \int \mathrm{d}\bar{c}\; \mathrm{d}c\, \tilde{S}[c_i u_i,\, \bar{c}_i v_i^\dagger,\, O] \tag{5.2}$$

is always smooth so long as the operator(s) O are smooth functions of the gauge field and $\tilde{S}$ is a smooth functional of the operators. We assume that u_i form an orthonormal basis of the +1-eigenspace of the operator $\hat{P}_-$, and that $\tilde{S}$ satisfies the usual chiral property: $\tilde{S}[X, Y^\dagger, O] = S[\hat{P}_- X, Y^\dagger, O]$ (we ignored the "chiral property" regarding the Y-fields here as those are assumed to be gauge field independent). Instead of e^S, we wrote $\tilde{S}$ to avoid dealing with possible logarithms in the proof which might cause unnecessary doubts. We have in mind that $\tilde{S}$ is, for example, given by the mirror fermion action averaged over the random unitary Higgs field(s) ϕ, the result of such averaging is a sum of multi-fermion terms, which are polynomials in terms of $c_i u_i$, and $\bar{c}_i v_i^\dagger$.

We say that Z is a "chiral partition function" in the general sense if it is defined by equation (5.2) with some $\tilde{S}$ that satisfies all the properties mentioned. We remind the reader that although the vectors u_i might be ill-defined at certain isolated points in the gauge field configuration space, they never diverge — they can not simply because they are unit vectors (see section 4.3.1 for example where we constructed the smooth measure $\mathcal{J}_\mu$ using them in the Wilson-line subspace with anomaly free contents). As a consequence, any "chiral partition function", being a grassmann integral defined with a smooth $\tilde{S}$, is always a finite function of the gauge field, even when evaluated infinitely close to the points where the basis vectors are ill-defined. More precisely stated, within any compact region in the gauge configuration space, the absolute value of any "chiral partition function" is bounded from above (with a fixed lattice size). We will loosely use the word "finite" in the following to describe this property.

The proof for the smoothness of Z is then really simple. One only needs to first notice that the variation of the action due to the variation of the operator O:

$$\tilde{S}'[X, Y, O] \equiv \frac{\delta \tilde{S}[X, Y, O]}{\delta O}\, \delta O\ , \tag{5.3}$$

is usually no longer chiral. However, if one defines:

$$\tilde{S}^{(1)}[X, Y, O] \equiv \tilde{S}'[\hat{P}_- X, Y, O] = \frac{\delta \tilde{S}[X', Y, O]}{\delta O}\, \delta O \Bigg|_{X' = \hat{P}_- X}\ , \tag{5.4}$$

it is manifestly chiral. It is easily verified that:

$$\int \mathrm{d}\bar{c}\mathrm{d}c\ \tilde{S}^{(1)}[c_i u_i, \bar{c}_i v_i^\dagger, O] = \int \mathrm{d}\bar{c}\mathrm{d}c\ \tilde{S}'[c_i u_i, \bar{c}_i v_i^\dagger, O]. \tag{5.5}$$

Furthermore $S^{(1)}$ is a smooth functional of O since the original action S and the operators are smooth as we assumed. Therefore, whenever Z is a "chiral partition function," Z', defined by:

$$Z' \equiv \int \mathrm{d}\bar{c}\mathrm{d}c\, \tilde{S}^{(1)}[c_i u_i,\, \bar{c}_i v_i^\dagger,\, O] = \int \mathrm{d}\bar{c}\mathrm{d}c\, \frac{\delta \tilde{S}[c_i u^i, \bar{c}_i v_i^\dagger, O]}{\delta O} \delta O, \tag{5.6}$$

is also "chiral" and thus finite as well. By the "splitting-theorem" of section 4.2, we have

$$\delta_\mu Z = Z \mathcal{J}_\mu + Z'. \tag{5.7}$$

Given that Z, Z' and $\mathcal{J}_\mu$ are all finite, we immediately know the first variation of Z, smooth or not, is at least finite.

We are now ready to claim, by applying the same logic iteratively, that given the assumptions listed above (smoothness of $\mathcal{J}_\mu$, $\tilde{S}$ and O), any "chiral partition function" Z (5.2) is smooth. This is because for $\forall n \in \mathbb{Z}$, the n-th derivative $Z^{(n)}$ can always be expressed as a polynomial in terms of some other "chiral partition functions" (which are always finite) and some smooth functions (the measure term $\mathcal{J}_\mu$ and its variations). This is certainly true when $n = 1$ as equation (5.7) says. Assuming the hypothesis holds true for some value of n, to prove that it remains true for $n+1$ is almost trivial. Just apply the above procedure on each "chiral partition function" appearing in the polynomial and recall that the derivative of any smooth function is still smooth. Hence, by induction this is true for any n. Because any "chiral partition function" is finite, so is $Z^{(n)}$. Therefore Z is smooth. Again, "finiteness" here means the function is bounded within any compact region in the gauge configuration space.[9]

Thus, the smoothness of the mirror partition function (and generating functional, with source terms for the mirror fields added) implies that an analytic or numerical result that would indicate the decoupling of the mirror sector (at strong Yukawa coupling, say, as in [41]) at vanishing gauge background would be expected to hold at least for "nearby" gauge backgrounds, e.g., in perturbation theory with respect to the gauge coupling. We think that this result clearly encourages further study of mirror-sector Yukawa-Higgs dynamics in anomaly free models.

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6. Addendum

In the first paragraph of section 5, we discussed the resolution to the paradox posed by the results of [41], in the situation that a dynamical gauge field is turned on. We argued that with a dynamical gauge field, the splitting of the fermions into different chiralities used in [41] is mathematically inconsistent, and that the numerical evidence found in [41] — using only the singular mirror partition function and indicating a complete decoupling of the mirror sector — can not be used to infer properties of the full theory with dynamical gauge fields. While this claim is plausible, it does not completely explain away the paradox,

[9] To be mathematically precise, we remind the reader that because the basis vectors are ill-defined at some isolated points in the gauge configuration space, the partition function Z, defined by (5.2), rigorously speaking is only defined everywhere away from those places. However, by showing the finiteness of all the derivatives evaluated infinitely close to those places, we have proved that those points are removable singularities of Z, namely near any one of those points x_0, $\lim_{x\to x_0} Z(x)$ exists, and is well-defined and finite. As long as we define $Z(x_0) = \lim_{x\to x_0} Z(x)$, Z is a smooth function on the entire gauge configuration space.

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as it leaves still unresolved questions if the gauge field is treated as an external background. In the interest of completeness we wish to briefly explain these questions here. We hope to return to their detailed study in the near future.

Since the obstruction to smoothness of the light-mirror split of the partition function is topological, given any gauge field background one can always find a choice of splitting that is smooth locally, in a small neighborhood in gauge field configuration space near the given point. If one treats this gauge field as a fixed external background only, some questions still remain. We point out an interesting observation here. Suppose $S = [X, Y^\dagger, O(A_\mu(x)]$ is any gauge invariant chiral action which satisfies:

$$S[X, Y^\dagger, O(A)] = S[\hat{P}(A)X, Y^\dagger, O(A)] = S[X, Y^\dagger P, O(A)]. \tag{6.1}$$

Here $\hat{P}(A) = \frac{1-\hat{\gamma}_5(A)}{2}$ and $P = \frac{1+\gamma_5}{2}$, $A_\mu(x)$ is the external gauge field and O represents all the operators appearing in the definition of S and typically depend on the gauge field A. With some orthonormal bases u_i and v_i, which are the appropriate eigenvectors of $\hat{P}$ and P respectively, with u_i chosen to be smooth with respect to A near, say, $A = 0$, the chiral partition function is defined as:

$$Z[A] = \int \mathrm{d}c_i \mathrm{d}\bar{c}_i \exp S[c_i u_i, \bar{c}_i v_i^\dagger, O(A)]. \tag{6.2}$$

We would like to calculate the polarization operator of A_μ at zero gauge field background, given by:

$$\Pi_{\mu\nu}(x,y) \equiv \left. \frac{\delta^2 \ln Z[A]}{\delta A_\mu(x) \delta A_\mu(y)} \right|_{A=0}. \tag{6.3}$$

Using the theorem of section 4.2, it is easily verified that $\Pi_{\mu\nu}$ splits into two parts.[10] The first part comes from the variations of the "measure term," due to the variation of $\ln Z$ caused by varying the eigenvectors u_i with respect to the gauge field. This part is not interesting to us. In particular, if we embed this chiral theory into any vector-like theory, this part is cancelled by the contributions from the fermions with opposite chirality and that of the Jacobian. The second part of $\Pi_{\mu\nu}(x,y)$ appears while one varies $\ln Z$ by varying the operators $O(A)$ with respect to the gauge field. This piece is physically more interesting. In the following discussion, we focus only on this "reduced" polarization:

$$\Pi'_{\mu\nu}(x,y) \equiv \left. \frac{\delta'^2 \ln Z[A]}{\delta' A_\mu(x) \delta' A_\mu(y)} \right|_{A=0}, \tag{6.4}$$

where δ' means variations with respect to A while keeping u_i fixed as constant vectors. Clearly $\Pi'_{\mu\nu}(x,y)$ can be expressed as some complicated fermion 2-point correlators in this theory and $\Pi'_{\mu\nu} = \Pi'_{\nu\mu}$. While evaluated on a translationally symmetrical background (e.g., $A = 0$), it depends on $|x - y|$ only.

[10] Notice that while computing the higher derivatives of $\ln Z$, one must follow the procedure outlined in section 5.

The divergence of this reduced 2-point function is easily calculated, since:

$$\sum_\mu \nabla^*_{\mu y} \frac{\delta' \ln Z[A]}{\delta' A_\mu(y) \delta' A_\nu(z)} = -\frac{\delta}{\delta\omega(y)} \sum_{\mu,x} \frac{\delta' \ln Z[A]}{\delta' A_\mu(x) \delta' A_\nu(z)} \nabla_{\mu x} \omega(x) \tag{6.5}$$

$$= -\frac{\delta'}{\delta' A_\nu(z)} \frac{\delta}{\delta\omega(x)} \sum_{\mu,y} \frac{\delta' \ln Z[A]}{\delta' A_\mu(y)} \nabla_{\mu y} \omega(y) \tag{6.6}$$

$$= \frac{\delta'}{\delta' A_\nu(z)} \frac{\delta}{\delta\omega(x)} \delta'_\omega \ln Z[A]. \tag{6.7}$$

Here $\delta'_\omega \ln Z[A]$ is the variation of $\ln Z[A]$ under the arbitrary gauge variation $A_\mu(x) \rightarrow A_\mu(x) - \nabla_\mu \omega(x)$, while keeping the basis vectors u_i fixed.

We have assumed that S is gauge invariant. Given this assumption, by equation (4.28), $\delta'_\omega \ln Z[A]$ is known to be exactly[11] $\frac{i}{2}\mathrm{Tr}\omega\hat{\gamma}_5$, completely independent to the details of S. It vanishes if and only if the anomaly cancellation condition is satisfied. Therefore in any anomalous chiral theory defined with projection operators P and $\hat{P}$ whose classical action is gauge invariant, there exists a fermion 2-point correlation function defined by (6.4), whose divergence is purely imaginary and proportional to $\delta \mathrm{tr}\hat{\gamma}_{5xx}/\delta A_\nu(y)$. Even though this expression is local, it is known that it is not the divergence of a local expression. Therefore, the fermion correlator, as part of the gauge field polarization operator, must contain a nonlocal contribution. The physical interpretation of this fact and its manifestation in the 1-0 model requires further studies. In particular, it will be interesting to see how it shows up in the numerical simulations. We hope to report on this subject in follow up work soon.

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[11] As usual, see section 3, this can be derived by first embedding the chiral theory into a vector-like theory defined by $S_{\mathrm{full}} = S[X, Y^\dagger, O] + Y^\dagger(1-P)D(1-\hat{P})X$ and then inferring the result from gauge invariance of the full theory.

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